

SLICE KNOTS WHICH BOUND KLEIN BOTTLES

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ABSTRACT. We investigate the properties of knots in \mathbb{S}^3 which bound Klein bottles, such that a pushoff of the knot has zero linking number with the knot, i.e. has *zero framing*. This is motivated by the many results in the literature regarding slice knots of genus one, for example, the existence of homologically essential zero self-linking simple closed curves on genus one Seifert surfaces for algebraically slice knots. Given a knot K bounding a Klein bottle F with zero framing, we show that J , the core of the orientation-preserving band in any disk-band form of F , has zero self-linking. We prove that such a K is slice in a $\mathbb{Z}[\frac{1}{2}]$ -homology \mathbb{B}^4 if and only if J is as well, a stronger result than what is currently known for genus one slice knots. As an application, we prove that given knots K and J and any odd integer p , the $(2, p)$ cables of K and J are $\mathbb{Z}[\frac{1}{2}]$ -concordant if and only if K and J are $\mathbb{Z}[\frac{1}{2}]$ -concordant. In particular, if the $(2, 1)$ -cable of a knot K is slice, K is slice in a $\mathbb{Z}[\frac{1}{2}]$ -homology ball.

1 INTRODUCTION

A *knot* is the image of a smooth embedding of an oriented $\mathbb{S}^1 \hookrightarrow \mathbb{S}^3 = \partial\mathbb{B}^4$. A knot is called *slice* if it bounds a smoothly embedded disk in \mathbb{B}^4 . The set of knots, modulo slice knots, under the connected sum operation forms an Abelian group called the *knot concordance group*, denoted by \mathcal{C} . In [26], Levine described a surjection from \mathcal{C} to $\mathbb{Z}^\infty \oplus (\mathbb{Z}/2\mathbb{Z})^\infty \oplus (\mathbb{Z}/2\mathbb{Z})^\infty$. Knots in the kernel of this map are said to be *algebraically slice*. The quotient of \mathcal{C} by algebraically slice knots is called the *algebraic knot concordance group*, denoted \mathcal{AC} .

It is a well-known fact that given any knot K , we can find an embedded oriented surface in \mathbb{S}^3 whose single boundary component is K . Such a surface is called a *Seifert surface*. Seifert surfaces give rise to a multitude of knot invariants, such as the *genus* of K , the minimum genus of a Seifert surface for K . There are many results in the literature about the properties of genus one knots, i.e. knots which bound punctured tori. These represent the simplest non-trivial class of Seifert surfaces. In [14] Gilmer showed that if a knot K is algebraically slice and bounds a punctured torus F , then, up to isotopy and orientation, there are exactly two homologically essential simple closed curves J_1 and J_2 on F with zero self-linking with respect to the Seifert form on F . This is an important result, since if one of these curves J_i is a slice knot, K must be slice as well. It is conjectured that the converse of the above statement is true, i.e. a genus one knot is smoothly slice if and only if there exists a homologically essential self-linking zero curve on the Seifert surface which is itself a smoothly slice knot [22, Strong Conjecture, pp. 226]. Much work has been done towards proving this result [8, 11, 13, 15]. Casson-Gordon theory can be used to show that at least

one of the curves J_i must satisfy some strong requirements on its algebraic concordance class, but it was recently shown that these fail to imply a vanishing signature function [16].

The motivation for our paper was to determine if similar results might be true for knots which bound punctured Klein bottles. Recall that the term *genus* for a connected, compact *non-orientable* surface is used to refer to the number of summands in its unique decomposition as a connected sum of real projective planes (with disks removed if necessary). In [4], Clark defined the *crosscap number* of a knot, denoted $c(K)$, to be the minimum genus of non-orientable surfaces bounded by K . This invariant is occasionally referred to as the *crosscap genus* or the *non-orientable genus* of K . $c(K)$ is a useful invariant since there are knots of arbitrarily large genus with $c(K) = 1$. A lot of work has been done on computing the crosscap numbers of certain families of knots, such as in [4, 20, 21, 31, 38, 39].

Knots with $c(K) = 1$ are completely classified by the following result of Clark:

Proposition 1.1 (Proposition 2.2. from [4]). *$c(K) = 1$ iff K is a $(2, n)$ cable knot.*

As a result, punctured Klein bottles represent the simplest non-trivial classes of non-orientable surfaces bounded by knots. Knots bounding Klein bottles were used in [19] to construct examples of topologically slice knots with nontrivial Alexander polynomials.

Suppose a knot K bounds a non-orientable surface F . If we define the *longitude* λ of K to be a pushoff in the direction of F , we see that λ bounds a non-orientable surface in the knot complement and therefore, has even linking number with the knot. In this paper we will assume, to parallel the orientable case, that $\text{lk}(K, \lambda) = 0$

Definition 2.1. *Let $K \subseteq \mathbb{S}^3$ be a knot and $F \subseteq \mathbb{S}^3$ be a non-orientable surface with $K = \partial F$. Let $N(K)$ be a regular neighborhood of K and the longitude $\lambda = F \cap \partial N(K)$. We define the framing of F to be $\text{lk}(K, \lambda)$, denoted $\mathcal{F}(F)$.*

The main result of our paper is the following:

Theorem 1.2. *If a slice knot K bounds a punctured Klein bottle F with $\mathcal{F}(F) = 0$, we can find a 2-sided homologically essential closed curve J embedded in F with self-linking zero which is slice in a $\mathbb{Z}[\frac{1}{2}]$ -homology ball and hence, rationally slice (i.e. slice in a \mathbb{Q} -homology \mathbb{B}^4).*

Rational concordance has been studied extensively and in great generality [1]. Being rationally slice is a strong condition since many classical concordance invariants secretly obstruct knots being \mathbb{Q} -concordant. For example, it is known that the Levine-Tristram signature function is zero for rationally slice knots [2]—the corresponding result about genus one slice knots is yet unknown, as shown in [16]. In addition, the τ -invariant of Ozsváth-Szabó and Rasmussen is known to be zero for rationally slice knots [33, 35]. These observations indicate that our result is stronger than what is currently known about genus one algebraically slice knots.

We will start by proving some general properties of non-orientable surfaces bounded by knots with zero framing, followed by the above theorem and other results relating to concordance. The tools developed will enable us to prove a surprising corollary about cable knots. We will use the notation $K_{(m,n)}$ to denote the (m, n) cable of a knot K . Details about the cabling operation can be found in any introductory knot theory textbook, such as [37,

Chapter 4D]. It can be easily shown that given concordant knots K and J , $K_{(m,n)}$ and $J_{(m,n)}$ are concordant for any choice of m and n . Using our results in Sections 2 and 3 we will prove the following partial converse:

Corollary 4.6. *Given knots K and J and any odd integer p , if the $(2, p)$ cables of K and J are concordant, K is concordant to J in a $\mathbb{Z}[\frac{1}{2}]$ -homology $\mathbb{S}^3 \times [0, 1]$. In particular, if $K_{(2,p)}$ is concordant to the $(2, p)$ torus knot, then K is slice in a $\mathbb{Z}[\frac{1}{2}]$ -homology \mathbb{B}^4 .*

This result is related to the recent work on studying whether satellite operations are injective on the smooth knot concordance group [9, 18], i.e. if two satellite knots on the same pattern knot are concordant, are the companion knots concordant? The (conjectured) smooth injectivity of the Whitehead doubling operator, for instance, has been studied for many years [25, Problem 1.38]. Corollary 4.6 has been generalized by Cochran, Davis and the author in [5].

1.1 NOTATION AND DEFINITIONS

We will work in the smooth category. Two knots $K_i \hookrightarrow \mathbb{S}^3 = \partial\mathbb{B}^4$, $i = 0, 1$, are said to be *concordant* if there exists a smooth proper embedding of an annulus into $\mathbb{S}^3 \times [0, 1]$ that restricts to K_i on each $\mathbb{S}^3 \times \{i\}$. A knot is called *slice*, if it is concordant to the unknot, or equivalently, if it is the boundary of a smooth embedding of a 2-disk in \mathbb{B}^4 .

There is a corresponding notion of knots being slice and concordant in spaces which look like \mathbb{B}^4 and $\mathbb{S}^3 \times [0, 1]$ with respect to homology with specified coefficients. Suppose $R \subseteq \mathbb{Q}$ is a non-zero subring. A space X is called an *R -homology Y* if $H_*(X; R) \cong H_*(Y; R)$. Knots K_0 and K_1 in \mathbb{S}^3 are said to be *R -concordant* if there exists a compact, oriented, smooth 4-manifold W such that W is an R -homology $\mathbb{S}^3 \times [0, 1]$, $\partial W = \mathbb{S}^3 \times \{0\} \sqcup -\mathbb{S}^3 \times \{1\}$, and there exists a smooth properly embedded annulus in W which restricts on its boundary to the given knots. We say that K is *R -slice* if it is R -concordant to the unknot, or equivalently if it bounds a smoothly embedded 2-disk in an R -homology 4-ball whose boundary is \mathbb{S}^3 . The set of knots modulo R -slice knots forms an Abelian group.

Two 3-manifolds M_1 and M_2 are said to be *homology cobordant* if there exists a 4-manifold W which is a smooth cobordism between M_1 and M_2 , such that $H_*(W, M_1) = 0 = H_*(W, M_2)$. For R as above, M_1 and M_2 are *R -homology cobordant* if there exists a W as above with the weaker requirement that $H_*(W, M_1; R) = 0 = H_*(W, M_2; R)$. For any knot K we will use the notation M_K to denote the zero-framed surgery on K .

A curve γ on a surface F is called *2-sided* if it has a regular neighborhood in F homeomorphic to an annulus, i.e. it has a trivial normal bundle. It is well known that if F is non-orientable, γ is orientation-preserving iff it is 2-sided. For a 2-sided curve there is well-defined notion of a pushoff γ^+ in a fixed direction: since we have a regular neighborhood of γ which is an annulus, γ^+ can be taken to be one of the boundary components of this annulus. The *self-linking* of γ is defined to be $\text{lk}(\gamma, \gamma^+)$.

We will also frequently require the ‘disk-band’ form of an embedded surface with boundary. We recall that given any embedding in \mathbb{S}^3 of a surface F with a single boundary component, there is an ambient isotopy of \mathbb{S}^3 taking F to the standard form of a disk with bands attached, wherein the bands may be twisted, linked or knotted, by collapsing towards

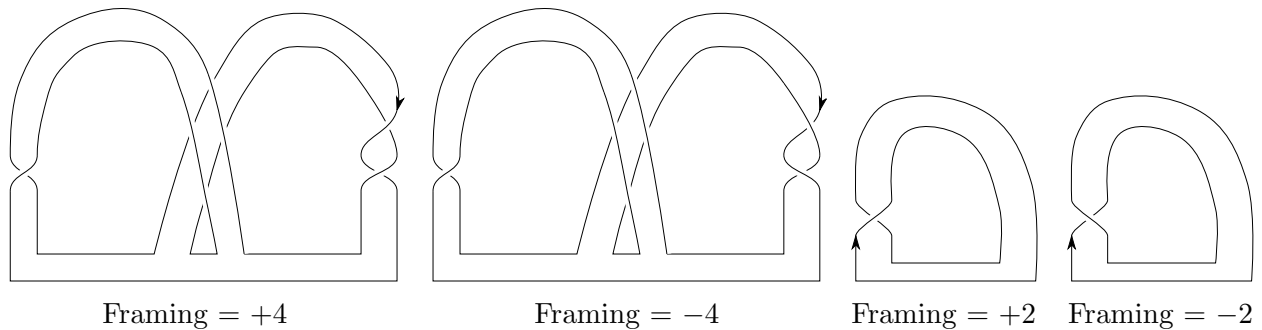


FIGURE 2.1.

the 1-skeleton. This process is described in [22, pp. 81]. We will additionally require that the disk-band form of a punctured Klein bottle contain an orientation preserving band, i.e. exactly one of the two bands in the disk band form has an odd number of half-twists.

2 PROPERTIES OF KNOTS BOUNDING KLEIN BOTTLES WITH ZERO FRAMING

We recall the following definition from Section 1:

Definition 2.1. *Let $K \subseteq \mathbb{S}^3$ be a knot and $F \subseteq \mathbb{S}^3$ be a non-orientable surface with $K = \partial F$. Let $N(K)$ be a regular neighborhood of K and the longitude $\lambda = F \cap \partial N(K)$. We define the framing of F to be $lk(K, \lambda)$, denoted $\mathcal{F}(F)$.*

Given an embedding of a surface F , we can first perform an ambient isotopy on \mathbb{S}^3 to get F in disk-band form. Given such an embedding, one can obtain $\mathcal{F}(F)$ by drawing a parallel to the boundary and computing the linking number. Such a calculation can be performed solely on the basis of the types and numbers of crossings of the bands, shown in Figure 2.4, and the twists in each band.

We notice that λ bounds a non-orientable surface in the complement of K , and therefore, $\mathcal{F}(F)$ is always an even number. In this paper we will further restrict $\mathcal{F}(F)$ to be zero to mirror the orientable case. We start by investigating some implications of the zero framing condition on any non-orientable surfaces which bound knots. First of all, it would be nice to know that this is possible:

Proposition 2.2. *Any knot K bounds some non-orientable surface (compact with a single boundary component smoothly embedded in \mathbb{S}^3) with zero framing.*

Proof. We know that any knot K bounds some such non-orientable surface, F , obtained using the checkerboard coloring of a diagram for K [4]. $\mathcal{F}(F)$ is an even number, which is additive under boundary connected sum of surfaces by the remarks above. If $\mathcal{F}(F) \equiv 0 \pmod{4}$, boundary connect sum F with as many copies of the Klein bottles shown in Figure 2.1 as needed, and if $\mathcal{F}(F) \equiv 2 \pmod{4}$ use the Möbius bands in Figure 2.1. This does not change the knot type of the boundary since the Klein bottles and Möbius bands in Figure 2.1 bound unknots. \square

Lemma 2.3. *Suppose a knot K bounds a non-orientable surface F with framing $\mathcal{F}(F)$. $\mathcal{F}(F) \equiv 2 \pmod{4}$ iff the genus of F is odd; $\mathcal{F}(F) \equiv 0 \pmod{4}$ iff the genus of F is even. In particular, if $\mathcal{F}(F) = 0$, F has even genus.*

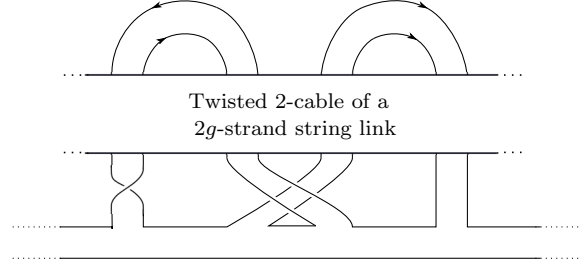


FIGURE 2.2. Disk band picture of a general non-orientable surface with even genus g .

Proof. For any knot K , if F is a surface (possibly non-orientable) with $\partial F = K$, there exists a non-singular symmetric bilinear form [17][27, Chapter 9]

$$\mathcal{G}_F : H_1(F) \times H_1(F) \rightarrow \mathbb{Z}$$

such that

$$\sigma(K) = \text{sign}(\mathcal{G}_F) - \frac{1}{2} \text{lk}(K, K^F)$$

where K^F is a parallel copy of K missing F and $\text{sign}(\mathcal{G}_F)$ is the signature of any matrix representing the bilinear form \mathcal{G}_F . We note that $\text{lk}(K, K^F) = \mathcal{F}(K)$ and is zero if F is orientable. Since \mathcal{G}_F is a non-singular bilinear form on $H_1(F)$, $\text{sign}(\mathcal{G}_F)$ is even exactly when $\dim(H_1(F))$ is even, i.e. F has even genus. Since $\sigma(K)$ is always even, $\frac{1}{2}\mathcal{F}(F)$ is even exactly when F has even genus. \square

Proposition 2.4. *If a knot K bounds a non-orientable surface F with zero framing, there is a 2-sided homologically essential closed curve embedded on F with zero self linking. In particular, the curve constructed is the Poincaré dual to $w_1(F)$, the first Stiefel-Whitney class of the tangent bundle of F .*

Proof. By Lemma 2.3, since $\mathcal{F}(F) = 0$, the genus of F is even. We perform an ambient isotopy of \mathbb{S}^3 to obtain F in disk-band form and then slide bands so that we get F as a

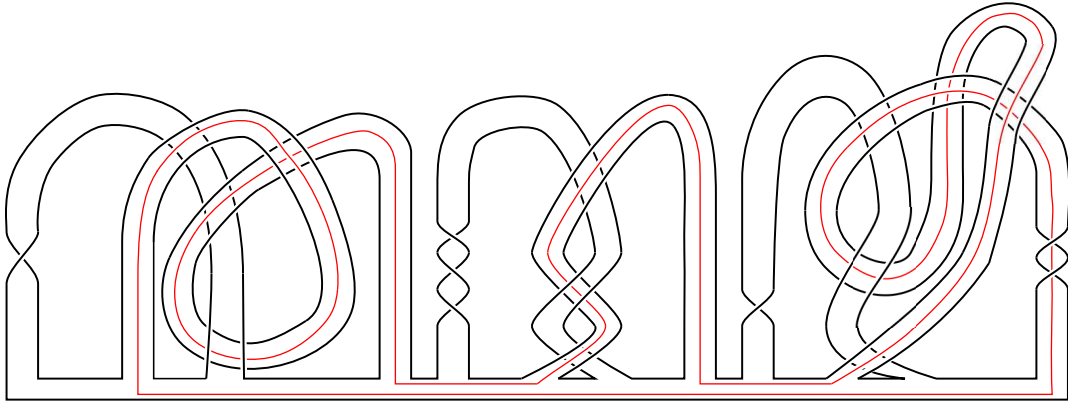
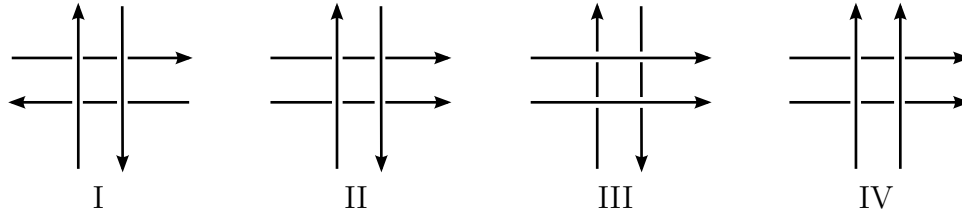


Figure 2.3. Curve of self-linking zero on a non-orientable surface bounding a knot with zero framing.


 FIGURE 2.4. Possible crossings of the bands in the disk-band form of F .

connected sum of punctured Klein bottles where each Klein bottle is of the form shown in Figure 2.2, i.e. each has one orientation-preserving band. Of course, the bands may interact with each other in ways other than crossings, as shown in Figure 2.3, and bands of different summands may also interact. As in the remarks at the beginning of this section, $\mathcal{F}(F)$ can be computed by considering the different types of crossings between bands and the twists within each band, in some projection for . Performing this calculation on each type of crossing in Figure 2.4 quickly shows that $\mathcal{F}(F)$ depends only on the crossings of type IV, each of which contributes ± 4 depending on the relative orientations of the crossing bands. Full twists of the orientation-preserving band can be deformed into crossings of type IV and also contribute ± 4 depending on the ‘handedness’ of the twist. Since the two edges of the orientation-reversing bands are oppositely oriented, twists in these bands do not contribute to $\mathcal{F}(F)$.

Consider γ , the curve which is the sum of the cores of the orientation preserving bands, as in shown in the example in Figure 2.3. $\text{lk}(\gamma, \gamma^+)$ can be calculated by considering only the crossings and twists of the orientation-preserving bands, which are exactly the crossings that contribute to $\mathcal{F}(F)$. In fact, $\mathcal{F}(F) = 4 \text{lk}(\gamma, \gamma^+)$. Therefore, $\mathcal{F}(F) = 0$ iff $\text{lk}(\gamma, \gamma^+) = 0$. By construction we see that this curve intersects each orientation-reversing curve on F transversely an odd number of times and each orientation-preserving curve an even number of times, which implies that it is the Poincaré dual of $w_1(F)$ and therefore, homologically essential. \square

Proposition 2.5. *The curve γ constructed in Proposition 2.4 when F is a punctured Klein bottle is unique upto orientation and isotopy.*

Proof. We see that the curve γ is 2-sided and non-separating when F is a Klein bottle. There are exactly four isotopy classes of unoriented homologically essential simple closed curves on a closed Klein bottle [30][34, Lemma 2.1]. Moreover, any two 2-sided non-separating simple closed curves are isotopic (as unoriented curves) on the closed Klein bottle. Henceforth, the proof is much like Gilmer’s proof of the corresponding fact about punctured tori in [15]. If we consider the isotopy on the closed Klein bottle, whenever the curve passes over the boundary component, we are effectively band-summing with the longitude of the Klein bottle. We check using a picture that band-summing γ with the longitude yields γ with the opposite orientation. \square

One reason for seeking curves of self-linking zero on a low genus Seifert surface is that one might perform surgery along it to reduce genus. The following result shows that the same is true for non-orientable surfaces.

Proposition 2.6. *Given a connected non-orientable surface F of genus g and a single boundary component, surgering along a non-separating 2-sided curve γ of zero self-linking, i.e. removing the annulus cobounded by two parallel copies of γ and gluing in two disks, results in a disk if $g = 2$. If the resulting surface is orientable, the genus is $\frac{g-2}{2}$; if the resulting surface is non-orientable, the genus is $g - 2$.*

Proof. We know that $\chi(F) = 1 - g$. Note that removing an annulus from F does not change the Euler characteristic, since $\chi(\text{annulus}) = \chi(\mathbb{S}^1) = 0$. Let F' the final surface with genus g' . We have

$$\begin{aligned}\chi(F') &= (1 - g) + \chi(2 \text{ disks}) - \chi(2 \text{ circles}) \\ &= (1 - g) + 2 - 0 \\ &= 3 - g\end{aligned}$$

Since γ is non-separating, F' is connected. If F' is non-orientable with genus g' , we have that $1 - g' = 3 - g \Rightarrow g' = g - 2$. If F' is orientable with genus g' , we have that $1 - 2g' = 3 - g \Rightarrow g' = \frac{g-2}{2}$. \square

Note that if surgery is performed on the curve γ dual to $w_1(F)$ constructed in Proposition 2.4 the resulting surface is necessarily orientable—since every orientation-reversing curve on the original surface intersected γ once, surgering along γ effectively removes all orientation-reversing curves from F .

The following are basic results for knots with crosscap number 2 which do not appear in the literature and will be used in the proof of Corollary 4.6.

Proposition 2.7. *Given any knots K and J , the composite knot $K_{(2,p)} \# J_{(2,-p)}$ bounds a Klein bottle F with zero framing. There is a disk-band form for F where the knot type of the orientation-preserving band is $K \# J$.*

Proof. $K_{(2,p)}$ and $J_{(2,-p)}$ bound Möbius bands with framing p and $-p$ respectively, by the definition of the cabling operation. Taking the boundary connected sum of the two Möbius bands gives us a Klein bottle with zero framing. However, while the obtained surface is in disk-band form it does not have an orientation-preserving band (yet). We can obtain one by sliding one of the erstwhile Möbius bands over the other. This results in an orientation preserving band whose core has the knot type $K \# J$. \square

The above proposition also implies that the $(2,1)$ cable of any knot K bounds a Klein bottle F with zero framing, where the knot type of the orientation-preserving band of F is K (by letting J be the unknot).

3 CONCORDANCE INVARIANTS

We recall the notation for infection on a knot, as described in [12]. We start with a *pattern knot* R , and an unknotted curve η in $\mathbb{S}^3 - R$ (the *axis of infection*). Since η is unknotted it bounds a disk. Tie all the strands of R passing through this disk into some knot J , the *infecting knot*. We obtain a knot as the result of infection and denote it by $R(\eta, J)$. It is easily seen that the above is a satellite operation.

Proposition 3.1. *If a knot K bounds a punctured Klein bottle F with zero framing, then K is smoothly concordant to a knot $R' = R(\eta, J)$, where R is a ribbon knot bounding a Klein bottle with zero framing, J is the knot type of the core of the orientation-preserving band of F given in disk-band form, and η is a curve as shown in Figure 3.1.*

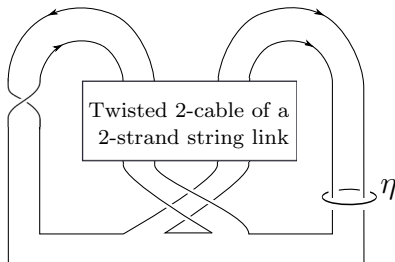


FIGURE 3.1. Knot bounding a Klein bottle with zero framing.

Proof. We will follow the proof of Proposition 1.7 in [7]. We isotope F into a disk-band form with an orientation-preserving band, as in the proof of Proposition 2.4. We also know from Proposition 2.4 that the core of the orientation-preserving band has zero self-linking. As a result, surgering along the core would give us a slice disk for the knot, as shown in Proposition 2.6. Let J denote the tangle whose closure is the knot type of the core of the orientation-preserving band of F . Notice that the orientation-preserving band can then be considered to be the (untwisted) 2-cable of J . Consider a curve η linking twice with the orientation-preserving band of F . It bounds a disk $E \subseteq \mathbb{S}^3$. If we thicken E we get the local picture shown in Figure 3.2. Replace the 2-cable of T by the 2-cable of $-J$, and call the resulting knot R . Notice that this results in a new Klein bottle, also with 0 framing. The knot R now bounds a Klein bottle where the knot type of the orientation-preserving band is $-J\#J$, which is ribbon. By Proposition 2.6, by surgering along the core of the orientation-preserving band, we see that the knot R is also ribbon.

Now consider the knot R' obtained from K by replacing T by the 2-cable of $-J\#J$. Note that $R' = R(\eta, J)$, i.e. the infection of R by J along the curve η , by the equivalence of the last two panels of Figure 3.2. Since the trivial tangle is smoothly concordant to $-J\#J$, their 2-cables are also smoothly concordant. By modifying the trivial concordance from K to itself

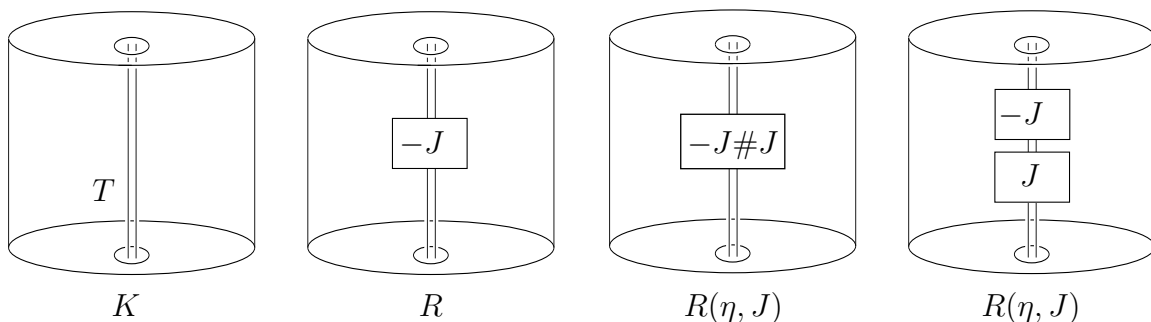


FIGURE 3.2. Proof of Proposition 3.1

by the tangle concordance between the 2-cables of the trivial tangle and $-J\#J$, we see that $R' = R(\eta, J)$ is smoothly concordant to K . \square

Recognizing that our knots are secretly infections provides a fair amount of information about certain knot invariants:

Proposition 3.2. *If a knot K bounds a punctured Klein bottle F with zero framing, and J is the knot type of the core of the orientation-preserving band of any disk-band form for F , we have the following results:*

- a. $\text{Arf}(K) = 0$.
- b. $\sigma_K(\omega) = \sigma_J(\omega^2)$, for all but finitely many ω .
- c. K has (ordinary) signature 0.
- d. $|\tau(K) - 2\tau(J)| \leq 4$.

Here $\sigma(\omega)$ denotes the Levine-Tristram signature function, and τ is the Floer homology invariant of Ozsváth-Szabó and Rasmussen [33, 35].

Proof. We know that $\text{Arf}(K) = 0$ iff $\Delta_K(-1) \equiv \pm 1 \pmod{8}$ for any knot K [32], where $\Delta_K(t)$ is the Alexander polynomial. On the other hand, since η has winding number 2,

$$\Delta_{R(\eta, J)}(t) = \Delta_R(t)\Delta_J(t^2)$$

that is, $\Delta_{R(\eta, J)}(-1) = \Delta_R(-1)\Delta_J(1)$. We know that $\Delta_J(1) = \pm 1$. Since the Arf invariant is a concordance invariant, R is ribbon, and $R(\eta, J)$ is smoothly concordant to K , we have that $\text{Arf}(R(\eta, J)) = \text{Arf}(K)$ and $\Delta_R(-1) \equiv \pm 1 \pmod{8}$. Part a. follows.

For Part b. we have from [28, 29] that

$$\sigma_{R(\eta, J)}(\omega) = \sigma_R(\omega) + \sigma_J(\omega^2)$$

since η has winding number 2, for all ω except the roots of the Alexander polynomials of K , J and R . Since R is ribbon, $\sigma_R(\omega)$ is the zero function, except at the roots of $\Delta_R(t)$. Part b. follows. Part c. follows as well by setting $\omega = -1$.

We have from Theorem 1.2 in [36] that

$$-n_+(R) - l \leq \tau(R(\eta, J)) - \tau(R) - l\tau(J) \leq n_+(R) + l$$

where $l = \text{lk}(R, \eta)$ and $n_+(R)$ is the least number of positive intersections between R and a disk bounded by η . In our case, we have $n_+(R) = l = 2$ and since R is smoothly slice, $\tau(R) = 0$. Also, since τ is an invariant of smooth concordance, $\tau(R(\eta, J)) = \tau(K)$. Therefore, $-4 \leq \tau(K) - 2\tau(J) \leq 4$, proving Part d. \square

Proposition 3.3. *If K is slice and bounds a Klein bottle F with zero framing, then J , the knot type of the core of the orientation preserving band in any disk-band form for F , is 2-torsion in the algebraic knot concordance group.*

Proof. Let \mathcal{AC} denote the algebraic knot concordance group, considered as the Witt group of nonsingular linking forms over certain torsion $\mathbb{Z}[t, t^{-1}]$ -modules [24]. Given a knot K , the corresponding element of \mathcal{AC} may be denoted by $(\mathcal{A}(K), \mathcal{B}l(K))$, where $\mathcal{A}(K)$ is the Alexander module of K and $\mathcal{B}l(K)$ is the Blanchfield linking form. That is, if we denote algebraic concordance class by $[\cdot]$, $[K] = (\mathcal{A}(K), \mathcal{B}l(K))$.

Consider the map $f : \mathcal{AC} \rightarrow \mathcal{AC}$, which takes $t \mapsto t^2$. Then $f([J]) = [K] = [R(\eta, J)]$, where R, η, J are as in Proposition 3.1. We know from [29] that

$$\mathcal{A}(R(\eta, J)) = \mathcal{A}_0(R) \oplus (\mathcal{A}_0(J) \otimes_{\mathbb{Z}[t, t^{-1}]} W)$$

where $W = \mathbb{Z}[t, t^{-1}]$ as a $\mathbb{Z}[t, t^{-1}]$ module, where t acts by $t \mapsto t^2$, since the winding number of η is 2. The map $t \mapsto t^2$ induces a similar transformation on the Blanchfield linking forms, that is, if $B_\cdot(t)$ is a matrix representing the Blanchfield linking form

$$B_{R(\eta, J)}(t) = B_R(t) \oplus B_J(t^2)$$

We denote this new Blanchfield form as $\mathcal{B}l(R(\eta, J)) = \mathcal{B}l(R) \oplus (\mathcal{B}l(J) \otimes_{\mathbb{Z}[t, t^{-1}]} W)$ where W is as above.

Since R is a ribbon knot, $(\mathcal{A}(R), \mathcal{B}l(R))$ is the zero Witt class in \mathcal{AC} . Therefore,

$$(\mathcal{A}(R(\eta, J)), \mathcal{B}l(R(\eta, J))) \cong (\mathcal{A}(J) \otimes_{\mathbb{Z}[t, t^{-1}]} W, \mathcal{B}l(J) \otimes_{\mathbb{Z}[t, t^{-1}]} W)$$

If the knot $R(\eta, J)$ is itself slice, we see that $(\mathcal{A}(J) \otimes_{\mathbb{Z}[t, t^{-1}]} W, \mathcal{B}l(J) \otimes_{\mathbb{Z}[t, t^{-1}]} W)$ is 0 in \mathcal{AC} , i.e. $f([J]) = 0$. But we know from [10, Proposition 2.1] (See also [3, Theorem 6]) that knots in the kernel of the map f induced by $t \mapsto t^2$ must be 2-torsion in \mathcal{AC} . \square

4 HOMOLOGY COBORDISM OF ZERO-SURGERY MANIFOLDS

Given a knot K , one frequently studies the associated 3-manifold, M_K , obtained by performing zero-framed surgery on K in \mathbb{S}^3 . Suppose the knots $K_0, K_1 \subseteq \mathbb{S}^3$ are concordant via an annulus $A \subseteq \mathbb{S}^3 \times [0, 1]$. By Alexander duality, the exterior of A is a \mathbb{Z} -homology cobordism between the exteriors of K_0 and K_1 . If we then adjoin a zero-framed $\mathbb{S}^1 \times \mathbb{D}^2 \times [0, 1]$ to the homology cobordism between exteriors, we get a homology cobordism between M_{K_0} and M_{K_1} . This observation has a converse when one of the knots is the unknot:

Proposition 4.1 (Proposition 1.2 from [6]). *Suppose K is any knot in \mathbb{S}^3 and U is the trivial knot. Then M_K is smoothly homology cobordant to M_U via a cobordism V whose π_1 is normally generated by a meridian of K if and only if K bounds a smoothly embedded disk in a smooth manifold that is homeomorphic to \mathbb{B}^4 .*

This result gives us a way of translating information about zero-surgery manifolds to information about concordance relationships between knots. Here, we will use a related result for R -homology cobordisms:

Proposition 4.2 (Proposition 1.5 from [6]). *Suppose K is any knot in \mathbb{S}^3 and $R \subseteq \mathbb{Q}$ is a non-zero subring. Let U denote the trivial knot. Then M_K is smoothly R -homology cobordant to M_U if and only if K is smoothly R -concordant to U i.e. K is smoothly R -slice.*

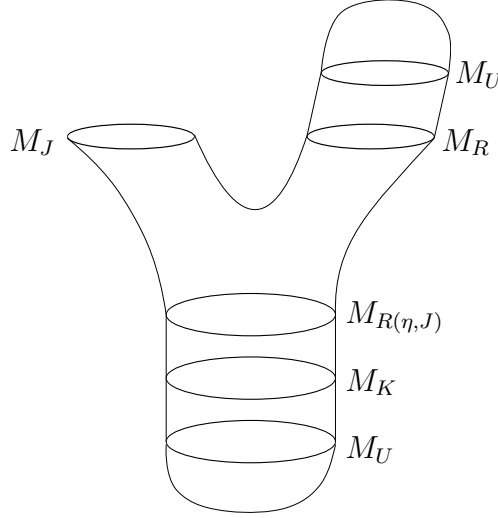


FIGURE 4.1. Proof of Theorem 4.4: Here $M_U \cong \mathbb{S}^1 \times \mathbb{S}^2$'s have been capped off by $\mathbb{S}^1 \times \mathbb{B}^3$'s.

In addition, recognizing that our knots are the result of infection allows us to use the following helpful theorem from [6]:

Theorem 4.3 (Theorem 2.1 from [6]). *Suppose R is a (smoothly) $\mathbb{Z}[\frac{1}{n}]$ -slice knot, and η is a curve with non-zero winding number n . Then, for any knot J , M_J is smoothly $\mathbb{Z}[\frac{1}{n}]$ -homology cobordant to $M_{R(\eta, J)}$.*

The proof of the following result is an extension of the proofs of the results above to our context.

Theorem 4.4. *Suppose the knot K is $\mathbb{Z}[\frac{1}{2}]$ -slice and bounds a punctured Klein bottle F with zero framing. Let J be the knot type of the orientation-preserving band in any disk-band form for F . Then J is smoothly $\mathbb{Z}[\frac{1}{2}]$ -slice, i.e. J bounds an embedded 2-disk in a 4-manifold \mathcal{B} which is a $\mathbb{Z}[\frac{1}{2}]$ -homology \mathbb{B}^4 .*

In addition, if K is smoothly slice, $\pi_1(\mathcal{B})$ is normally generated by a single element (the meridian of K), the meridian of J is mapped to twice the generator of H_1 of the slice disk complement in \mathcal{B} , and the homology groups of \mathcal{B} are as follows:

- $H_1(\mathcal{B}; \mathbb{Z}) = H_2(\mathcal{B}; \mathbb{Z}) = \mathbb{Z}/2$
- $H_3(\mathcal{B}; \mathbb{Z}) = H_4(\mathcal{B}; \mathbb{Z}) = 0$

Proof. By Proposition 3.1, K is smoothly concordant to $R(\eta, J)$, where R is a ribbon knot and J is as required. By the remarks at the beginning of this section, this gives us a \mathbb{Z} -homology cobordism between M_K and $M_{R(\eta, J)}$. Since R is smoothly slice and η has winding number 2, Theorem 4.3 gives us a smooth $\mathbb{Z}[\frac{1}{2}]$ -homology cobordism between $M_{R(\eta, J)}$ and M_J . Since K is $\mathbb{Z}[\frac{1}{2}]$ -slice, we have a $\mathbb{Z}[\frac{1}{2}]$ -homology cobordism between M_K and M_U , where U is the unknot, by Proposition 4.2. By stacking the various cobordisms as in Figure 4.1, we obtain that M_J is smoothly $\mathbb{Z}[\frac{1}{2}]$ -homology cobordant to M_U , and by Proposition 4.2, J is smoothly $\mathbb{Z}[\frac{1}{2}]$ -slice. This completes the proof of the first part of this theorem.

To complete the proof, we need to take a closer look at the cobordism promised by Theorem 4.3. Following the construction in [6], we have a cobordism between $M_R \sqcup M_J$ and $M_{R(\eta,J)}$, obtained as follows. Start with $M_J \times [0, 1]$ and $M_R \times [0, 1]$. Let $N(\eta)$ be a regular neighborhood of η in M_R . We identify $N(\eta) \times \{1\} \subseteq M_R \times \{1\}$ with the surgery solid torus in $M_J \times [0, 1]$ such that a parallel pushoff of η identifies with the meridian of K . The resulting 4-manifold that boundary $M_J \sqcup M_R \sqcup -M_{R(\eta,J)}$. In addition, we have that R is smoothly slice, and therefore, M_R is homology cobordant to $M_U \cong \mathbb{S}^1 \times \mathbb{S}^2$, which can be capped off by $\mathbb{S}^1 \times \mathbb{B}^3$. This gives us the cobordism between $M_{R(\eta,J)}$ and M_J claimed in Theorem 4.3 (the top half of Figure 4.1).

In this case, K is slice, and therefore, we can cap off the lower boundary component of the cobordism, as shown in Figure 4.1, by attaching the homology cobordisms between M_K and $M_{R(\eta,J)}$, and between M_K and M_U . Finally we cap off $M_U \cong \mathbb{S}^1 \times \mathbb{S}^2$ by $\mathbb{S}^1 \times \mathbb{B}^3$. This gives us a 4-manifold bounded by M_J . We add a zero-framed 2-handle to M_J along the meridian of J to finally obtain the manifold \mathcal{B} with $\partial\mathcal{B} = \mathbb{S}^3$, in which J bounds a smoothly embedded disk, as desired.

The cobordism between M_R , M_J and $M_{R(\eta,J)}$ deformation retracts to $M_{R(\eta,J)} \cup \eta \times \mathbb{B}^2$, so up to homotopy, we obtain the cobordism by adding a 2-cell and a 3-cell. The 2-cell is added along λ_J , the longitude of J . Moreover, $\pi_1(M_{R(\eta,J)})$ is normally generated by the meridian of $R(\eta, J)$. The fundamental group of each of the other cobordisms is normally generated by the meridian of the relevant knot and the 2-handle added at the final stage kills off μ_J , the meridian of the knot J . Therefore, we have:

$$\begin{aligned} \pi_1(\mathcal{B}) &= \left(\langle \langle \mu_K \rangle \rangle / \langle \langle \lambda_J \rangle \rangle \right) / \left(\langle \langle \mu_J \rangle \rangle / \langle \langle \lambda_J \rangle \rangle \right) \\ &= \langle \langle \mu_K \rangle \rangle / \langle \langle \mu_J \rangle \rangle \end{aligned}$$

In particular, $\pi_1(\mathcal{B})$ is normally generated by μ_K . Note that, in homology, $\mu_K^2 = \mu_J$ and hence, $H_1(\mathcal{B}; \mathbb{Z}) \cong \mathbb{Z}/2$. Since $\tilde{H}_i(\mathcal{B}; \mathbb{Z}[\frac{1}{2}]) = 0$, $\tilde{H}_i(\mathcal{B}; \mathbb{Z})$ is 2-torsion. We can recover all the other homology groups using the Universal Coefficient Theorem and Poincaré-Lefschetz Duality. All the homology groups below are with \mathbb{Z} coefficients:

$$\begin{aligned} \mathbb{Z}/2 \cong H_1(\mathcal{B}) &\cong H^3(\mathcal{B}, \partial\mathcal{B}) \cong \text{Hom}(H_3(\mathcal{B}, \partial\mathcal{B}), \mathbb{Z}) \oplus \text{Ext}(H_2(\mathcal{B}, \partial\mathcal{B}), \mathbb{Z}) \\ &\Rightarrow \text{Ext}(H_2(\mathcal{B}, \partial\mathcal{B}), \mathbb{Z}) \cong \mathbb{Z}/2 \\ &\Rightarrow \text{Torsion}(H_2(\mathcal{B}, \partial\mathcal{B})) \cong \mathbb{Z}/2 \end{aligned}$$

Recall that $\partial\mathcal{B} = \mathbb{S}^3$. Therefore, using the homology exact sequence for a pair, we have:

$$\begin{aligned} 0 \rightarrow H_2(\mathcal{B}) &\xrightarrow{\cong} H_2(\mathcal{B}, \partial\mathcal{B}) \rightarrow 0 \rightarrow H_1(\mathcal{B}) \xrightarrow{\cong} H_1(\mathcal{B}, \partial\mathcal{B}) \rightarrow 0 \\ 0 \leftarrow H^2(\mathcal{B}) &\xleftarrow{\cong} H^2(\mathcal{B}, \partial\mathcal{B}) \leftarrow 0 \leftarrow H^1(\mathcal{B}) \xleftarrow{\cong} H^1(\mathcal{B}, \partial\mathcal{B}) \leftarrow 0 \end{aligned}$$

$$H_3(\mathcal{B}) \cong H^1(\mathcal{B}, \partial\mathcal{B}) \cong H^1(\mathcal{B}) \cong \text{Hom}(H_1(\mathcal{B}), \mathbb{Z}) \cong 0$$

Since $H_2(\mathcal{B})$ is 2-torsion and $H_2(\mathcal{B}) \cong H_2(\mathcal{B}, \partial\mathcal{B})$ and $\text{Torsion}(H_2(\mathcal{B}, \partial\mathcal{B})) \cong \mathbb{Z}/2$, $H_2(\mathcal{B}) \cong \mathbb{Z}/2$.

We note that the slice disk Δ_J bounded by J in the construction above is the 2-handle added at the last stage and therefore, μ_J is mapped to twice the generator of $H_1(\mathcal{B} - \Delta_J) \cong \mathbb{Z} = \langle \mu_K \rangle$. \square

We should note that the condition of the meridian of J mapping to twice the generator of the slice disk complement is related to the notion of being *weakly* rationally slice [23]. In addition, we know that if J is $\mathbb{Z}[\frac{1}{2}]$ -slice, so is K , and therefore, we have actually proved that K is $\mathbb{Z}[\frac{1}{2}]$ -slice if and only if J is $\mathbb{Z}[\frac{1}{2}]$ -slice. Moreover, in conjunction with the remark following the proof of Proposition 2.7, we have now proved:

Corollary 4.5. *For a knot K if the $(2,1)$ cable is slice, or even just $\mathbb{Z}[\frac{1}{2}]$ -slice, then K is $\mathbb{Z}[\frac{1}{2}]$ -slice.*

We are also now able to prove the following:

Corollary 4.6. *Given knots K and J , if $K_{(2,p)}$ is $\mathbb{Z}[\frac{1}{2}]$ -concordant to $J_{(2,p)}$, then K is $\mathbb{Z}[\frac{1}{2}]$ -concordant to J . In particular, if $K_{(2,p)}$ is concordant to the $(2,p)$ torus knot, then K is $\mathbb{Z}[\frac{1}{2}]$ -slice.*

Proof. First we note that $-(J_{(2,p)}) = (-J)_{(2,-p)}$. We know from Proposition 2.7 that $K_{(2,p)} \# - (J_{(2,p)}) = K_{(2,p)} \# (-J)_{(2,-p)}$ bounds a Klein bottle with zero framing, where we may consider $K \# - J$ to be the knot type of the orientation-preserving band. Since $K_{(2,p)}$ is $\mathbb{Z}[\frac{1}{2}]$ -concordant to $J_{(2,p)}$, $K_{(2,p)} \# - (J_{(2,p)})$ is $\mathbb{Z}[\frac{1}{2}]$ -slice, and we are in the situation of Theorem 4.4. Therefore, $K \# - J$ is $\mathbb{Z}[\frac{1}{2}]$ -slice, and so K is $\mathbb{Z}[\frac{1}{2}]$ -concordant to J . \square

REFERENCES

- [1] J. C. Cha. The structure of the rational concordance group of knots. *Mem. Amer. Math. Soc.*, 189(885):x+95, 2007.
- [2] J. C. Cha and K. H. Ko. Signatures of links in rational homology spheres. *Topology*, 41(6):1161–1182, 2002.
- [3] J. C. Cha, C. Livingston, and D. Ruberman. Algebraic and Heegaard-Floer invariants of knots with slice Bing doubles. *Math. Proc. Cambridge Philos. Soc.*, 144(2):403–410, 2008.
- [4] B. E. Clark. Crosscaps and knots. *International Journal of Mathematics and Mathematical Sciences*, 1:113–124, 1978.
- [5] T. D. Cochran, C. D. Davis, and A. Ray. Injectivity of satellite operators in knot concordance. Preprint: <http://arxiv.org/abs/1205.5058>, 2012.
- [6] T. D. Cochran, B. D. Franklin, M. Hedden, and P. D. Horn. Knot concordance and homology cobordism. *Proceedings of the American Mathematical Society: to appear*, 2011.
- [7] T. D. Cochran, S. Friedl, and P. Teichner. New constructions of slice links. *Commentarii Mathematici Helvetici*, 84:617–638, 2009.
- [8] T. D. Cochran, S. Harvey, and C. Leidy. Derivatives of knots and second-order signatures. *Algebr. Geom. Topol.*, 10(2):739–787, 2010.
- [9] T. D. Cochran, S. Harvey, and C. Leidy. Primary decomposition and the fractal nature of knot concordance. *Math. Ann.*, 351(2):443–508, 2011.
- [10] T. D. Cochran and K. E. Orr. Not all links are concordant to boundary links. *Ann. of Math. (2)*, 138(3):519–554, 1993.
- [11] T. D. Cochran, K. E. Orr, and P. Teichner. Knot concordance, Whitney towers and L^2 -signatures. *Ann. of Math. (2)*, 157(2):433–519, 2003.
- [12] T. D. Cochran, K. E. Orr, and P. Teichner. Structure in the classical knot concordance group. *Comment. Math. Helv.*, 79(1):105–123, 2004.

- [13] D. Cooper. *Signatures of surfaces with applications to knot and link cobordism*. PhD thesis, University of Warwick, 1982.
- [14] P. M. Gilmer. Slice knots in S^3 . *Quart. J. Math. Oxford Ser. (2)*, 34(135):305–322, 1983.
- [15] P. M. Gilmer. Classical knot and link concordance. *Commentarii Mathematici Helvetici*, 68:1–19, 1993.
- [16] P. M. Gilmer and C. Livingston. On surgery curves for genus one slice knots. Preprint: <http://arxiv.org/abs/1109.1518>, 2011.
- [17] C. M. Gordon and R. A. Litherland. On the signature of a link. *Inventiones Mathematicae*, 47:53–69, 1978.
- [18] M. Hedden and P. Kirk. Instantons, Concordance, and Whitehead doubling. Preprint: <http://arxiv.org/abs/1009.5361>, 2010.
- [19] M. Hedden, C. Livingston, and D. Ruberman. Topologically slice knots with nontrivial Alexander polynomial. Preprint: <http://arxiv.org/abs/1001.1538>, 2010.
- [20] M. Hirasawa and M. Teragaito. Crosscap numbers of 2-bridge knots. *Topology*, 45(3):513–530, 2006.
- [21] K. Ichihara and S. Mizushima. Crosscap numbers of pretzel knots. *Topology and its Applications*, 157(1):193–201, 2010.
- [22] L. H. Kauffman. *On knots*, volume 115 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1987.
- [23] A. Kawauchi. Rational-slice knots via strongly negative-amphicheiral knots. *Commun. Math. Res.*, 25(2):177–192, 2009.
- [24] C. Kearton. Cobordism of knots and Blanchfield duality. *J. London Math. Soc. (2)*, 10(4):406–408, 1975.
- [25] R. Kirby, editor. *Problems in low-dimensional topology*, volume 2 of *AMS/IP Stud. Adv. Math.* Amer. Math. Soc., Providence, RI, 1997.
- [26] J. P. Levine. Invariants of knot cobordism. *Invent. Math.* 8 (1969), 98–110; addendum, *ibid.*, 8:355, 1969.
- [27] W. B. R. Lickorish. *An introduction to knot theory*, volume 175 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1997.
- [28] R. A. Litherland. Signatures of iterated torus knots. In *Topology of low-dimensional manifolds (Proceedings Second Sussex Conference, Chelwood Gate, 1977)*, volume 722 of *Lecture Notes in Mathematics*, pages 71–84. Springer, Berlin, 1979.
- [29] C. Livingston and P. Melvin. Abelian invariants of satellite knots. In *Geometry and Topology*, volume 1167 of *Lecture Notes in Mathematics*, chapter 13, pages 217–227. Springer Berlin / Heidelberg, 1985.
- [30] W. H. Meeks, III. Representing codimension-one homology classes on closed nonorientable manifolds by submanifolds. *Illinois J. Math.*, 23(2):199–210, 1979.
- [31] H. Murakami and A. Yasuhara. Crosscap number of a knot. *Pacific Journal of Mathematics*, 171(1):261–273, 1995.
- [32] K. Murasugi. The Arf invariant for knot types. *Proc. Amer. Math. Soc.*, 21:69–72, 1969.
- [33] P. Ozsváth and Z. Szabó. Knot Floer homology and the four-ball genus. *Geom. Topol.*, 7:615–639, 2003.
- [34] T. M. Price. Homeomorphisms of quaternion space and projective planes in four space. *J. Austral. Math. Soc. Ser. A*, 23(1):112–128, 1977.
- [35] J. A. Rasmussen. *Floer homology and knot complements*. PhD thesis, Harvard University, 2003.
- [36] L. P. Roberts. Some bounds for the knot Floer τ -invariant of satellite knots. *Algebraic and Geometric Topology*, 12(1):449–467, 2012.
- [37] D. Rolfsen. *Knots and links*, volume 7 of *Mathematics Lecture Series*. Publish or Perish Inc., Houston, TX, 1990. Corrected reprint of the 1976 original.
- [38] M. Teragaito. Creating Klein bottles by surgery on knots. *Journal of Knot Theory and its Ramifications*, 10(5):781–794, 2001. Knots in Hellas '98, Vol. 3 (Delphi).
- [39] M. Teragaito. Crosscap numbers of torus knots. *Topology and its Applications*, 138(1-3):219–238, 2004.